Finally, similar curves were drawn for System T—mean photographic R.A., and the results were decidedly more consistent than for C.

23. From these investigations, which have been tested and varied in every way that can be thought of, the following results emerge:—

System T has a fairly uniform magnitude-equation of about $+0^s\cdot015$ per magnitude throughout list I. and the first half of list II.; for the second half there is little material.

System L and the independent adopted systems have a magnitude-equation of about $+0^s\cdot020$ for list I. and $+0^s\cdot015$ for list II.

System C has a magnitude-equation of about $-0^s\cdot017$ for about half list I. and of $-0^s\cdot002$ for the remainder and for list II.

It seems to me, therefore, that so far as magnitude-equation is concerned System T is little better than System L, while System C is worse, because it changes so suddenly. From this point of view there would be little advantage in adopting either T or C as the standard system for the reduction of all the photographs.

My sincere acknowledgments are due to Miss Julia Bell, who has carried out very skilfully the numerical work summarised in this paper. A large portion of the expense has been borne by a grant from the Royal Society Government Grant Fund.

*Cambridge Observatory:*
1906 *June* 7.

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## *On some Points connected with the Determination of Orbits.*
By H. C. Plummer, M.A.

1. The difference between the circular measure and the sine of an angle is an expression which occurs in several of the most important formulæ relating to elliptic motion. When the angle is large there is no difficulty in calculating this difference accurately, but when it is small ordinary logarithmic tables will not give the required accuracy without the help of some special device. Hence auxiliary tables have been published in a variety of forms, according to the purpose for which they are designed. It is possible to restrict the compass of such tables by making them merely supplementary to the ordinary logarithms by the use of some artifice. Of this nature is Tietjen's * formula, which may be expressed in the form

$$\epsilon - \sin \epsilon = \tfrac{4}{3}\,\mathrm{B} \sin^{3}\tfrac{1}{2}\,\epsilon\,(\sec \tfrac{1}{4}\,\epsilon)^{2\cdot4} \qquad \ldots \qquad \ldots \qquad (1)$$

* *A.N.* 1463; also Watson's *Theoretical Astronomy*, p. 343.

for which a very small table, giving log B as far as $\epsilon=60°$, is necessary. But the use of any special table can be avoided. Thus I have pointed out elsewhere * that the function

$$\tfrac{1}{6}\,\epsilon^3/\,(\sec \tfrac{1}{12}\,\epsilon)^{14\cdot4} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

is a remarkably close approximation to the value of $\epsilon-\sin\epsilon$; indeed, over the range of $\epsilon$ from 0° to 70° the difference between the logarithms of the two expressions does not exceed 2 in the seventh place. A greater error may be caused by the occurrence of the power 14·4. It may be noted that if $\epsilon$ is less than 36° the logarithm of the required secant can be obtained from Schrön's tables (as the difference between S and T) to eight places. Within this range at least the approximation may be considered practically perfect.

2. A function of such frequent occurrence that it has been made the subject of special tables is

$$Q(\epsilon) = \tfrac{3}{4}(\epsilon - \sin\,\epsilon)\,/\sin^3\tfrac{1}{2}\epsilon.$$

This can be expressed by Tietjen's formula thus:

$$Q(\epsilon) = B(\sec \tfrac{1}{4}\epsilon)^{2\cdot4} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

or by using the approximate expression (2)

$$Q(\epsilon) = (\tfrac{1}{2}\epsilon/\sin \tfrac{1}{2}\epsilon)^3\,(\sec \tfrac{1}{12}\epsilon)^{-14\cdot4} \quad \ldots \quad \ldots \quad (4)$$

provided $\epsilon<70°$. Now the nature of the function $Q(\epsilon)$, which is unity when $\epsilon=0°$, suggests that it may be represented very approximately by the form $(\sec b\epsilon)^a$, $a$ and $b$ being two constants so chosen as to give the best possible result. For practical purposes, however, $b$ must be a very simple factor. This limitation precludes a quite satisfactory approximation: no great improvement can be made on the trigonometrical factor in (3). But the case is different if we seek to represent Q not by a single term $(\sec b\epsilon)^a$, but by a product of two such terms. This is shown by considering the logarithmic expansions of the functions involved.

3. In the first place, if $\mu$ is the modulus of common logarithms,

$$\log\,(\epsilon-\sin\,\epsilon)-3\,\log\,\epsilon+\log\,6$$
$$=\mu\left[-\tfrac{1}{20}\epsilon^2-\tfrac{1}{16800}\epsilon^4+\tfrac{1}{756000}\epsilon^6+\tfrac{89}{3104640000}\epsilon^8+\ldots\right] \quad (5)$$

We have also

$$\log \sin x = \log x - \mu\Sigma\, B_{2i-1}\frac{2^{2i}}{2i\,\lfloor 2i}\,x^{2i} \quad \ldots \quad \ldots \quad (6)$$

$$\log \sec x = \mu\Sigma\, B_{2i-1}\frac{2^{2i}(2^{2i}-1)}{2i\,\lfloor 2i}x^{2i} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

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* *A.N* 3883

where $B_1$, $B_3$, ... are Bernoulli's numbers, of which the first four are $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$ and $\frac{1}{30}$. By means of (5) and (6) we find

$$\log Q(\epsilon)=\mu\left[\tfrac{3}{40}\epsilon^2+\tfrac{11}{11200}\epsilon^4+\tfrac{1}{56000}\epsilon^6+\tfrac{701}{2069760000}\epsilon^8+\ldots\right] \quad (8)$$

The object is now to combine two series of the form (7) in such a way as to reproduce the first terms of (8). Theoretically an attempt might be made to obtain the first four terms; but if simple multiples of $\epsilon$ only are admitted, a less perfect agreement must suffice. It is indeed remarkable enough that with the multiples $\frac{1}{4}$ and $\frac{1}{6}$ of $\epsilon$ three terms can be made to coincide. Thus the application of (7) gives

$$\begin{aligned}&\tfrac{24576}{7000}\log\sec\tfrac{1}{4}\epsilon-\tfrac{17496}{7000}\log\sec\tfrac{1}{6}\epsilon\\ &=\mu\left[\tfrac{3}{40}\epsilon^2+\tfrac{11}{11200}\epsilon^4+\tfrac{1}{56000}\epsilon^6+\tfrac{17}{48384000}\epsilon^8+\ldots\right]\\ &=\log Q(\epsilon)-\tfrac{59}{4656960000}\mu\epsilon^8-\ldots\end{aligned}$$

The outstanding part of the fourth term amounts to 1 in the seventh place when $\epsilon=82°$ and the succeeding terms are still smaller. Practically throughout the first quadrant we may write

$$\begin{aligned}\log Q(\epsilon)&=\tfrac{24576}{7000}\log\sec\tfrac{1}{4}\epsilon-\tfrac{17496}{7000}\log\sec\tfrac{1}{6}\epsilon\\ &=[0{\cdot}5454132]\log\sec\tfrac{1}{4}\epsilon-[0{\cdot}3978408]\log\sec\tfrac{1}{6}\epsilon \quad \ldots \quad (9)\end{aligned}$$

The range is greater than is actually necessary, since Q can be calculated directly when $\epsilon$ has a fairly large value.

4. As an example of the use of such formulæ let us consider the method of Gauss for determining an orbit when two heliocentric positions are known. This involves the solution of the equations

$$y^2=m/(l+\sin^2\tfrac{1}{2}g) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$
$$y^3-y^2=m(2g-\sin 2g)/\sin^3 g=\tfrac{4}{3}mQ(2g) \quad \ldots \quad \ldots \quad (11)$$

where $y$ is the ratio of the sector to the triangle, $2g$ is the difference of the eccentric anomalies, and $l$ and $m$ are given quantities. These equations can be solved by trial without using special tables such as have been given by Gauss. The natural course is to choose an approximate value of $2g$ (in the absence of more precise knowledge, $2f$, the difference of the true anomalies, may be taken) and to deduce $y$ by means of (11). Then (10) will give an improved value of $g$, with which the process can be repeated.

Let $y=1/c\sinh\alpha$. Then, by (11),

$$4mQc^3\sinh^3\alpha+3c\sinh\alpha-3=0,$$

which can be compared with

$$4\sinh^3\alpha+3\sinh\alpha-\sinh 3\alpha=0,$$

giving $$mQc^2 = 1, \quad c \sinh 3a = 3.$$

Now if $$e^{3a} = \cot \tfrac{1}{2}\beta, \quad e^{a} = \cot \tfrac{1}{2}\gamma$$
$$\sinh 3a = \cot \beta, \ \sinh a = \cot \gamma.$$

For (10) we may write

$$\sin^2 \tfrac{1}{2}g = l \tan^2 \delta, \ y^2 = m \cos^2 \delta / l.$$

Then the calculation from an assumed value of $g$ to a better approximation is reduced to the following system of equations :—

$$\left.\begin{aligned} \cot \beta &= 3m^{\frac{1}{2}}Q^{\frac{1}{2}} \\ \tan^3 \tfrac{1}{2}\gamma &= \tan \tfrac{1}{2}\beta \\ \cos \delta &= l^{\frac{1}{2}}Q^{\frac{1}{2}} \tan \gamma \\ \sin \tfrac{1}{2}g &= l^{\frac{1}{2}} \tan \delta \\ \text{with} \qquad y &= m^{\frac{1}{2}}l^{-\frac{1}{2}} \cos \delta \end{aligned}\right\} \quad \ldots \quad \ldots \quad (12)$$

the last step being made only after satisfactory values of $g$ and $\delta$ are known.

5. The application of the method will be made clearer by a numerical example. The following * is chosen : $t'-t$, the interval of time, $=$ 100 days, $\log r = 0{\cdot}221\ 6050$, $\log r' = 0{\cdot}209\ 9050$, $2f$, the angle between the two radii, $= 44^\circ 25' 48''{\cdot}00$ ; whence

$$m = k^2(t'-t)^2/8 \cos^3 f(rr')^{\frac{3}{2}} = [9{\cdot}021\ 2961]$$
$$l = (r+r')/4 \cos f(rr')^{\frac{1}{2}} - \tfrac{1}{2} = [8{\cdot}603\ 5663]$$

The complete calculation by means of Bremiker's 6-figure logarithms is given below (see p. 495).

Now if, as in the given calculation, one assumed value $x_1$ leads to a closer approximation $x_2$, and $x_2$ similarly leads to $x_3$, the approximate correction to $x_3$ is

$$x - x_3 = (x_3 - x_2)^2 / \{(x_2 - x_1) - (x_3 - x_2)\} \quad \ldots \quad \ldots \quad (13)$$

which, applied to the successive values of $\tfrac{1}{2}g$, gives

$$\tfrac{1}{2}g - \tfrac{1}{2}g_3 = (63''{\cdot}8)^2 \div 3658'' = +1''{\cdot}11.$$

The calculation of $y$ can now be completed thus :—

| | | | |
|---|---|---|---|
| $\tfrac{1}{2}g$ | $12^\circ\ 5'\ 18''{\cdot}1$ | $\delta$ | $46^\circ\ 16'\ 6''{\cdot}2$ |
| L. $\sin \tfrac{1}{2}g$ | 9·321018 | L. $\cos \delta$ | 9·839654 |
| $\tfrac{1}{2}$ L. $l$ | 9·301783 | $\tfrac{1}{2}$ L. $m$ | 9·510648 |
| | | $-\tfrac{1}{2}$ L. $l$ | 0·698217 |
| L. $\tan \delta$ | 0·019235 | | |
| | | L. $y$ | 0·048519 |

* For the same example worked by Gauss' special tables see Bauschinger's *Tafeln zur Theoretischen Astronomie*, p. 26.

*Calculation referred to on p. 494.*

| | | | | | |
|---|---|---|---|---|---|
| $2g$ | 44° 25′ 48″·0 | 48° 25′ 23″·2 | $\beta$ | 45° 9′ 20″·0 | 45° 1′ 56″·2 |
| $\frac{1}{2}g$ | 11 6 27·0 | 12 6 20·8 | $\frac{1}{2}\beta$ | 22 34 40·0 | 22 30 58·1 |
| $\frac{1}{3}g$ | 7 24 18·0 | 8 4 13·9 | | | |
| | | | L. tan $\frac{1}{2}\beta$ | 9·618890 | 9·617571 |
| L. sec $\frac{1}{2}g$ | 0·008213 | 0·009766 | L. tan $\frac{1}{2}\gamma$ | 9·872963 | 9·872524 |
| L. sec $\frac{1}{3}g$ | 0·003637 | 0·004323 | | | |
| | | | $\frac{1}{2}\gamma$ | 36° 44′ 13″·6 | 36° 42′ 33″·6 |
| LL. sec $\frac{1}{3}g$ | 7·560743 | 7·635785 | $\gamma$ | 73 28 27·2 | 73 25 7·2 |
| const. | 0·397841 | 0·397841 | | | |
| L. (1) | 7·958584 | 8·033626 | L. tan $\gamma$ | 0·527678 | 0·526136 |
| | | | $\frac{1}{2}$ L. $l$ | 9·301783 | 9·301783 |
| LL. sec $\frac{1}{2}g$ | 7·914502 | 7·989717 | $\frac{1}{2}$ L. Q | 0·009872 | 0·011741 |
| const. | 0·545413 | 0·545413 | | | |
| L. (2) | 8·459915 | 8·535130 | L. cos $\delta$ | 9·839333 | 8·839660 |
| (2) | 0·028835 | 0·034287 | $\delta$ | 46° 18′ 32″·3 | 46° 16′ 3″·6 |
| (1) | 0·009090 | 0·010805 | | | |
| | | | L. tan $\delta$ | 0·019851 | 0·019224 |
| L. Q | 0·019745 | 0·023482 | $\frac{1}{2}$ L. $l$ | 9·301783 | 9·301783 |
| L. 9 | 0·954243 | 0·954243 | | | |
| L. $m$ | 9·021296 | 9·021296 | L. sin $\frac{1}{2}g$ | 9·321634 | 9·321007 |
| sum | 9·995284 | 9·999021 | $\frac{1}{2}g$ | 12° 6′ 20″·8 | 12° 5′ 17″·0 |
| L. cot $\beta$ | 9·997642 | 9·999511 | | | |

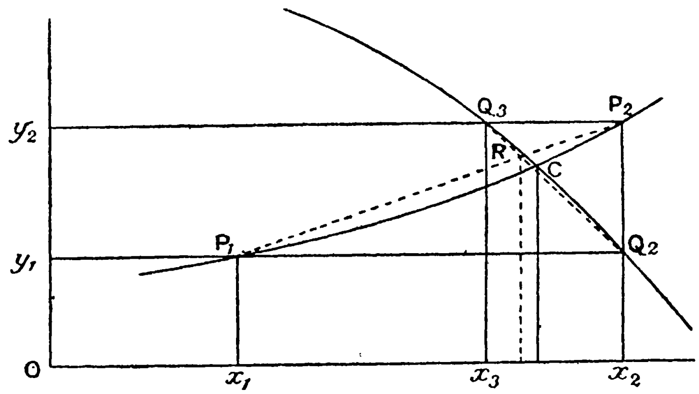
The value found by 7-figure logarithms and Gauss' tables is log $y = 0{\cdot}0485191$. The accuracy of the result is naturally greater than can be expected in general from a 6-figure computation, but the above example is merely illustrative of the method, and a more accurate calculation is therefore unnecessary.

6. It may be remarked that a still rougher computation may be of use as a preliminary step. Thus 4-figure logarithms are enough to give $\frac{1}{2}g$ in this case with an error not greater than 1′. But for the effective application of (13) it is necessary to make the two successive calculations with nearly the same order of accuracy. This formula, which is quite general, can easily be proved on the assumption that only the first powers of the errors committed are of importance. The general problem is to solve the equations

$$y = p(x),\ y = q(x),$$

where $p$ and $q$ are functions which involve algebraic or transcendental forms, by assuming a value $x_1$ and deducing $x_2$, which in turn is made a starting-point to obtain $x_3$. The two equations

may be represented graphically by the curves $P_1CP_2$ and $Q_2CQ_3$, the coordinates of whose intersection C are to be found. The method may then be translated thus: (1) with the assumed abscissa $x_1$ find the ordinate $y_1$ of $P_1$, (2) with this ordinate find the abscissa $x_2$ of $Q_2$ on the second curve, (3) with the abscissa $x_2$ find the ordinate $y_2$ of $P_2$ on the first curve, (4) with the ordinate $y_2$ find the abscissa $x_3$ of $Q_3$ on the second curve. It is then clear that the meaning of the correction given by (13) is to give the point R in which the straight lines $P_1P_2$ and $Q_2Q_3$ intersect. This graphical representation shows clearly the general nature of the process, and suggests in an interesting way the conditions on which its effectiveness depends. These conditions involve the slope and curvature of the lines, and are well illustrated by the special problem discussed above.




7. The equations (10) and (11), which are due to Gauss, are closely related to Lambert's theorem for elliptic motion, which is expressed by the familiar equation

$$a^{-\frac{3}{2}}k(t'-t) = (\epsilon - \sin\epsilon) - (\delta - \sin\delta) \quad \ldots \quad \ldots \quad (14)$$

where

$$\sin^2 \tfrac{1}{2}\epsilon = \frac{r+r'+c}{4a}, \ \sin^2 \tfrac{1}{2}\delta = \frac{r+r'-c}{4a} \quad \ldots \quad \ldots \quad (15)$$

I have shown * that when the eccentricity is moderate and the interval between the observations is not unreasonably large, this system of equations is capable of a convenient solution and gives a practical method of determining the elements of an orbit. The usefulness of the theorem has been more generally recognised in cases of large (nearly parabolic) eccentricity. Now it is obvious that a direct calculation of $t'-t$ will not be accurate when the mean distance $a$ is large and consequently $\epsilon$ and $\delta$ are small, especially if the chord $c$ is also small in comparison with the radii. Hence Marth † has derived elaborate expansions in series and calculated extended tables from the results. The value of these tables is well known. Yet, though they are

* *M.N.* vol. lxiii. p. 147. † *A.N.* 1557–1560.

doubtless convenient, they are not indispensable. It seems worthy of notice that complete accuracy can be attained by the ordinary forms of calculation.

8. With any assumed value of $a$, $\epsilon$ and $\delta$ can be calculated by (15); moreover, the difference of these equations gives

$$\sin\tfrac{1}{2}(\epsilon-\delta)\sin\tfrac{1}{2}(\epsilon+\delta) = c/2a \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

whence the difference $\epsilon-\delta$ can also be calculated with all necessary accuracy. Now since

$$2\sin\tfrac{1}{2}(\epsilon-\delta)-(\sin\epsilon-\sin\delta) = 2\sin\tfrac{1}{2}(\epsilon-\delta)\{1-\cos\tfrac{1}{2}(\epsilon+\delta)\}$$
$$= c/a\,.\tan\tfrac{1}{4}(\epsilon+\delta)$$

the equation (14) can now be written

$$k(t'-t) = a^{\frac{1}{2}}c\tan\tfrac{1}{4}(\epsilon+\delta)$$
$$+\ 2a^{\frac{3}{2}}\{\tfrac{1}{2}(\epsilon-\delta)-\sin\tfrac{1}{2}(\epsilon-\delta)\} \quad \ldots \quad (17)$$

We now have a sum of two *positive* terms, each of which can be calculated without difficulty. By (16) this becomes

$$k(t'-t) = a^{\frac{1}{2}}c\tan\tfrac{1}{4}(\epsilon+\delta)+\tfrac{1}{24}a^{-\frac{3}{2}}c^3\sin^{-3}\tfrac{1}{2}(\epsilon+\delta).\ \mathrm{A} \quad \ldots \quad (18)$$

where

$$\mathrm{A} = 6\{\tfrac{1}{2}(\epsilon-\delta)-\sin\tfrac{1}{2}(\epsilon-\delta)\}/\sin^3\tfrac{1}{2}(\epsilon-\delta)$$

and tends to the value 1 when $\epsilon-\delta$ is small. It is always possible to calculate log A with the help of (2) or (9). Frequently the second term in (18) is small compared with the first term, and A is not required with great accuracy. An approximation which may be useful can be obtained by the method of § 3. This is

$$\log \mathrm{A} = 3{\cdot}6\log\sec\tfrac{1}{4}(\epsilon-\delta) \quad \ldots \quad \ldots \quad (19)$$

which exceeds the true value by 1 in the seventh place when $\epsilon-\delta=12°$, and will therefore in general be good enough in the case of an ellipse which is nearly parabolic.

9. It is of interest to notice the corresponding form of Euler's equation for parabolic motion. We have $a$ infinite and $\delta=\epsilon=0$, but at the same time the finite limits

$$a\epsilon^2 = r+r'+c, \qquad a\delta^2 = r+r'-c.$$

Since now A = 1, (17) becomes

$$k(t'-t) = \tfrac{1}{4}c\mathrm{R}+\tfrac{1}{3}c^3/\mathrm{R}^3 \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where

$$\mathrm{R} = (r+r'+c)^{\frac{1}{2}}+(r+r'-c)^{\frac{1}{2}}.$$

This, unlike the ordinary form, is suitable for the calculation of $t'-t$ however small $c$ may be, although not so convenient as the trigonometrical transformation employed by Encke.*

* *Berl. Jahrbuch*, 1833, p. 268

10. The numerical application of (18) is in practice very simple and convenient. For example, let $r = 1{\cdot}5$, $r' = 1{\cdot}51$, and $c = 0{\cdot}15$, and let us calculate the time-interval corresponding to these values * of $a$ : 1·55, 10, 400 and infinity. We obtain the following results :—

| $a$. | Period. Years. | $\frac{1}{2}\epsilon$. ° ′ ″ | $\frac{1}{2}\delta$. ° ′ ″ | $t'-t$. Days. |
|---|---|---|---|---|
| 1·55 | 1·93 | 45 33 16·26 | 42 46 47·55 | 10·549300 |
| 10 | 31·6 | 16 19 26·55 | 15 30 33·49 | 7·865279 |
| 400 | 8000 | 2 32 49·630 | 2 25 23·236 | 7·570711 |
| ∞ | ∞ | ... | ... | 7·563420 |

In all these cases the second term on the right of (18), or (20) in the case of the parabola, could be calculated by 4-figure logarithms. Further, the value of log A was 0·0005 in the first case, and negligible for the other two elliptic orbits. The form (18) will always give accurate results, and the advantage which it offers becomes most conspicuous in those cases which have been considered to present the greatest difficulty.

*University Observatory, Oxford:*
1906 *June* 4.

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*Errors of Tabular Place of Jupiter, from Photographs taken with the Astrographic 13-inch Refractor of the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer-Royal.*)

In communicating the results of measures of photographs of the sixth and seventh satellites of *Jupiter* it was pointed out that in deducing the position angle and distance the error of the tabular place of *Jupiter* had been neglected.

To eliminate this error, and also that arising from any systematic error of the catalogues employed, a series of photographs of *Jupiter* was taken with the astrographic 13-inch refractor, with exposures only just long enough to give good measurable images of the known stars, *i.e.* $30^s$.

In all ten photographs, taken between 1905 November 3 and 1906 February 15, have been selected and measured. Four images each of *Jupiter* and of about twelve stars were measured on each plate, the positions of the stars being derived from the *Astronomische Gesellschaft Catalogue* (Berlin zones). The deduced positions of *Jupiter* are thus affected by any systematic error of the catalogue and the error due to twenty-five years' unknown proper motion of the stars; but, as the positions of the satellites, deduced in the same manner, are affected by the same errors, it

* For two methods of calculating $t'-t$ for the case $a = 10$ see Bauschinger's *Tafeln*, pp. 32, 33.